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Strings, Dipoles and Fuzzy Spheres

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Abstract

I discuss a scaling limit, where open strings in the WZW-model behave as dipoles with charges confined to a spherical brane and projected to the lowest Landau level. Then I show how the joining and splitting interactions of these dipoles are naturally described using the fuzzy sphere algebra.

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1 Introduction

The idea that space coordinates might be noncommutative is due to Heisenberg. Later Peierls showed that the coordinates of particles in strong magnetic fields become noncommutative upon projection to the lowest Landau level (LLL). However, this realization of noncommutativity is phenomenological and led Snyder to propose a *fundamental* quantization of space coordinates. In more recent times this proposal was successfully implemented for gauge theories on noncommutative spaces such as the noncommutative plane, torus [1] or sphere [2].

This begs the question as to why such deformations of ordinary quantum field gauge theory exist and whether this is related to the original phenomenological realization of noncommutativity as a projection to the LLL in strong magnetic fields. Indeed, in the flat space case the low energy effective description of strings in a strong NS-NS background is given by noncommutative gauge theory [3, 4, 6, 7, 8]. For reviews see [9, 10, 11]. It was emphasized in [12] that at low energy strings behave as dipoles with the charges in the LLL and the joining and separation interaction of these dipoles is naturally described by gauge theory on the noncommutative plane [13, 12]. For further detailed studies see also [14, 15, 16, 17].

In this paper I explore a similar limit for open strings on group manifolds. I will concentrate on the SU(2) group but most of the analysis can be generalized to an arbitrary group G. Note that $AdS_3 \times S^3 \times T^4$ is an exact string theory background, thus the SU(2) WZW-model corresponds to the S^3 factor and can be embedded in string theory. Then, the D2-branes on which strings end are ordinary two spheres. I show that there exists a scaling limit similar to the one in flat space and then explore how the fuzzy sphere algebra arises naturally. The resulting noncommutative gauge theory on a fuzzy sphere was first derived by Alekseev, Recknagel and Schomerus in [18]. However, the derivation uses conformal field theory results and does not reveal the simple geometric picture of interacting dipoles which immediately leads to the fuzzy sphere algebra. Our derivation identifies the states of the string at low energy with the wave function of the dipole in the LLL. Furthermore, we also show that a remarkable formula used to describe multiplication on the fuzzy sphere has the direct physical interpretation as an interaction vertex.

The paper is organized as follows. Section 2 contains a review of D-branes and open strings in the WZW-model. In section 3, I introduce a scaling limit and obtain the low

energy action. I quantize the reduced action in section 4 and show how the joining and splitting interaction is naturally described by the fuzzy sphere algebra. Finally, in the appendix I show how Kaluza-Klein reduction can be used to obtain the quantum states of the particle in a monopole magnetic field.

2 D-branes in the WZW-model

In this section we review D2-branes and open strings in the $SU(2)$ level k WZW-model. This subject has a long history starting with Ishibashi and Cardy [19, 20] who studied consistent boundary states in the WZW-model using current algebra techniques. In [21] an action approach was used to show that D2-branes with magnetic flux and lying on conjugacy classes give a consistent extension of the WZW-model to open strings. It was realized in [22] that the Cardy states are nothing but the D2-brane conjugacy classes introduced in [21]. The relation of these D-branes to fuzzy spheres was first noted in [18] and the noncommutative gauge field theory describing the low energy excitations of open strings was written down in [23]. Finally in [24], using the Born-Infeld action it was shown that these spherical D2-branes are stable. In this section I will review the WZW action of open strings on group manifolds following [21], but using a different notation in order to agree with the standard conventions.

2.1 Closed strings

Before I discuss D-branes and open strings let me first consider closed strings. Just as the action of a particle coupled to a gauge field A_μ is not gauge invariant, the string coupled to the $B_{\mu\nu}$ field does not have a gauge invariant action. To obtain a gauge invariant action one has to consider a closed path for the particle or a closed world sheet for the string. In the Minkowski formulation one can take the difference between two world sheets with the same initial and final string configurations. Let $g : \Sigma \to G$ denote an arbitrary map from the world sheet to the target group manifold G connecting the initial and final string configurations. Also choose a fixed reference map $g : \Sigma_0 \to G$ connecting the same initial and final string configurations[a]. We can now form a gauge

[a]The use of the same letter g is intentional, as g denotes a single function defined on $\Sigma - \Sigma_0$

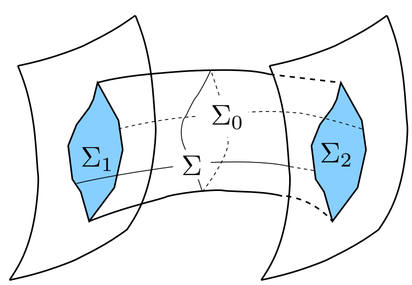
invariant combination from the WZW actions $\mathcal{S}_\Sigma$ and $\mathcal{S}_{\Sigma_0}$ given by

$$\mathcal{S}_\Sigma - \mathcal{S}_{\Sigma_0} = \frac{k}{8\pi}\oint_{\Sigma-\Sigma_0} d^2\sigma\, \mathrm{Tr}(g^{-1}\partial^\mu g g^{-1}\partial_\mu g) + \frac{k}{12\pi}\int_{\mathcal{B}} \mathrm{Tr}\left[(\tilde{g}^{-1}d\tilde{g})^{\wedge 3}\right] \ . \tag{1}$$

Here $\mathcal{B}$ is any 3-dimensional manifold such that its boundary $\partial\mathcal{B}$ satisfies $\partial\mathcal{B} = \Sigma - \Sigma_0$, and $\tilde{g}$ is an arbitrary map $\tilde{g} : \mathcal{B} \to G$ such that it coincides with g when restricted to the boundary $\Sigma - \Sigma_0$. Note that since the second homology group $H_2(G)$ is trivial it is always possible to construct the map $\tilde{g}$. The second term is the nonlinear sigma model and the last term is the Wess-Zumino action. The only ambiguity in (1) is in the choice of the manifold $\mathcal{B}$ and of the map $\tilde{g}$. The difference between two such choices in (1) is the integral of the 3-form $H = \frac{k}{12\pi}\mathrm{Tr}\left[(\tilde{g}^{-1}d\tilde{g})^{\wedge 3}\right]$ on a closed 3-dimensional sub-manifold of G. If the level k is an integer the ambiguity in the action is a multiple of 2π which is allowed quantum mechanically [25]. Mathematically, this is the statement that $H \in \mathrm{H}^3(G, \mathbb{Z})$ i.e. H is an integral cohomology cycle.

2.2 Open strings

In the open string case the endpoints of the string are constrained to live on D-branes which are sub-manifolds of G. We further assume that there exist $U(1)$ gauge fields localized on the D-branes. In this case $\Sigma - \Sigma_0$ has two closed boundaries Δ_1 and Δ_2 localized on the D-branes. If we further assume that the D-branes are simply connected Δ_1 and Δ_2 are contractable to a point. Let Σ_1 and Σ_2 denote 2-manifolds with boundaries Δ_1 and Δ_2. Then $\Sigma - \Sigma_0 + \Sigma_1 - \Sigma_2$ is a closed 2-dimensional surface, as shown in the figure below.



Let $\mathcal{B}$ denote any 3-dimensional manifold such that $\partial\mathcal{B} = \Sigma - \Sigma_0 + \Sigma_1 - \Sigma_2$ and let B denote a two form such that $dB = H$. If F is the field strength of the $U(1)$ gauge field

in the D-brane we can define the open version of (1)

$$\begin{aligned} \mathcal{S}_\Sigma - \mathcal{S}_{\Sigma_0} &= \frac{k}{8\pi}\int_{\Sigma-\Sigma_0} d^2\sigma\, \mathrm{Tr}(g^{-1}\partial^\mu g g^{-1}\partial_\mu g) + \\ & \frac{k}{12\pi}\int_{\mathcal{B}} \mathrm{Tr}\left[(\tilde{g}^{-1}d\tilde{g})^{\wedge 3}\right] + \\ & \int_{\Sigma_1}(F-B) - \int_{\Sigma_2}(F-B) \quad . \end{aligned} \tag{2}$$

Note that (2) is manifestly gauge invariant since the gauge transformation $B' = B + d\Lambda$ is accompanied by $A' = A + \Lambda$, thus $F - B$ is gauge invariant. There are further ambiguities in (2) besides the ones found in (1) related to the choice of Σ_1 and Σ_2. Again quantum mechanics allows for a multiple of 2π ambiguity in the action and this implies that the integral on F on any closed two manifold must be an integer. In conclusion, we have found that the following topological quantizations

$$H \in \mathrm{H}^3(G,\mathbb{Z}) \ , \quad F \in \mathrm{H}^2(G,\mathbb{Z})$$

are the necessary and sufficient conditions for a consistent path integral formulation of the open WZW-model.

2.3 Born-Infeld stability analysis

If D2-branes exist in the SU(2) level k WZW-model their ground state must be S^2 by symmetry. But there are no nontrivial cycles with this topology in $G = \mathrm{SU}(2)$ therefore there must exist a dynamical mechanism that stabilizes the branes. Indeed Bachas, Douglas and Schweigert used the Born-Infeld effective action to show that the S^2 branes are stable [24]. Let us briefly review their analysis.

The metric in the $SU(2)$ level k WZW-model is the standard S_3 metric

$$ds^2 = k\alpha' \left[d\psi^2 + \sin^2\!\psi\, (d\phi^2 + \sin^2\!\theta\, d\phi^2)\right] \ ,$$

and the NS-NS field strength is up to normalization the volume 3-form on S_3 given by

$$H = dB = \frac{k}{\pi}\sin^2\!\psi\, \sin\theta\ d\psi\, d\theta\, d\phi \ .$$

Consider a spherical D2-brane located at some fixed ψ and carrying magnetic flux $m \in \mathbb{Z}$ satisfying $0 < m < k$. The uniform gauge field is

$$F = dA = -\frac{m}{2}\sin\theta\, d\theta\, d\phi \ . \tag{3}$$

The energy of such a configuration (to lowest order in α') obtained using the Born-Infeld action is given by

$$\begin{aligned} E_m(\psi) &= T_{(2)} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{\det(\hat{G} + 2\pi\alpha'(\hat{B} + F))} \\ &= 4\pi k\alpha' T_{(2)} \left(\sin^4\psi + (\psi - \frac{\sin 2\psi}{2} - \frac{\pi m}{k})^2 \right)^{1/2} , \end{aligned}$$

where $\hat{G}$ and $\hat{B}$ are the induced metric and NS-NS 2-form and $T_{(2)}$ is the D2-brane tension. There is a unique minimum of the energy for $0 < m < k$ given by

$$\psi_m = \frac{\pi m}{k} .$$

Furthermore, in [24] it was also shown that these spherically symmetric configurations are stable against small fluctuations. A quadratic expansion around the spherical D-brane contains only positive mass terms except for three zero modes corresponding to translations of locations of the center of mass away from pole at $\psi = 0$.

3 Scaling limit

In this section I consider the scaling limit obtained by taking k to infinity and holding m fixed. First I will write down the action for the $SU(2)$ level k WZW-model describing open strings ending on the D2-brane located at ψ_m in a fixed gauge

$$\begin{aligned} \mathcal{S} = & \int d\tau \int_{-l/2}^{l/2} d\sigma \quad \frac{k}{2} \Big\{ [\dot{\psi}^2 + \sin^2\psi\, (\dot{\phi}^2 + \sin^2\theta\, \dot{\phi}^2)] - && (4) \\ & [\psi'^{\,2} + \sin^2\psi\, (\phi'^{\,2} + \sin^2\theta\, \phi'^{\,2})] + && (5) \\ & \left(\psi - \frac{\sin(2\psi)}{2} \right) \sin\theta\, (\dot{\theta}\, \phi' - \theta'\, \dot{\phi}) \Big\} + && (6) \\ & \int d\tau\, \frac{m}{2} \left(\cos\theta_+\, \dot{\phi}_+ - \cos\theta_-\, \dot{\phi}_- \right) . && (7) \end{aligned}$$

The terms (4) and (5) are the nonlinear σ-model part of the action. The term (6) is a local parameterization of the WZ-action with the B-field nonsingular at $\psi = 0$. The final term (7) gives the coupling of the two ends of the string to the gauge field on the D2-brane (3).

First do the following rescaling

$$\left\{ \begin{array}{rcl} \psi &=& \frac{2\pi}{k} r , \\ \tau &=& \frac{k}{2\pi} t . \end{array} \right. \qquad (8)$$

To motivate this rescaling first note that large k is the semi-classical limit and we expect string excitations to decouple. Since the brane is located at $\psi_m = \frac{\pi m}{k}$ and the string ψ coordinate must be of order ψ_m it is useful to introduce a rescaled coordinate r which will be finite. The stable brane labeled by m is now located at

$$r_m = \frac{m}{2} \ .$$

The rescaled time t is introduced so that the low energy states of the string are also of order one. Upon inserting (8) into the WZW action, the terms (4) and (6) scale as k^{-2} while the terms (5) and (7) scale as k^0. Thus as $k \to \infty$ the WZW action reduces to

$$\begin{aligned} \mathcal{S} \;\; = \;\; & \int dt \, \frac{m}{2} \left(\cos(\theta_+)\dot{\phi}_+ - \cos(\theta_-)\dot{\phi}_-\right) - & (9) \\ & \int dt \int_{-l/2}^{l/2} d\sigma \, \frac{1}{2} \left[r'^{\,2} + r^2 \left(\theta'^{\,2} + \sin^2\theta \, \phi'^{\,2} \right) \right] . & (10) \end{aligned}$$

Note that the quadratic kinetic term (4) has disappeared and the action is already in Hamiltonian form. However now only the endpoints of the string have conjugate canonical momenta. The bulk of the string coordinates are auxiliary fields and can be integrated out. For large k the S_3 sphere becomes very large and the strings see a flat metric. That is why I used r to denote the rescaled coordinate ψ. In Cartesian coordinates, the equations of motion derived from (10) are just $x_a'' = 0$, thus the string is just a straight line segment connecting two points on the $D2$-brane. Integrating these equations and plugging the solution back in (10) we obtain the Hamiltonian

$$\mathcal{H} = \frac{1}{2l} \Delta^2 \ , \tag{11}$$

where Δ denotes the length of the string.

To review, I have found that the large k limit of the WZW open strings with the rescaling (8) behave like dipoles whose charges are connected by an elastic string of string constant l^{-1}. Furthermore, no mass term is present for the charges therefore upon quantization only the LLL will be present.

4 The fuzzy sphere algebra

In this section I will present a brief review of the fuzzy sphere algebra. Consider the following angular momentum truncation of functions of the S^2 sphere

$$f(\theta,\phi) = \sum_{l=0}^{m} \sum_{p=-l}^{l} C_p^l Y_l^p(\theta,\phi) \ . \tag{12}$$

The space of truncated functions is denoted S^2_{m+1} and could be used to obtain a finite number of degrees of freedom for a field theory on S^2. However, the set of truncated functions do not form an algebra with respect to the usual function multiplication so it appears difficult to write any interaction terms without higher angular momenta resurfacing.

Fortunately a modified multiplication exists which makes the set of truncated functions (12) into an algebra. It is defined as follows. First recall that the spherical harmonics are traceless homogeneous polynomials in the normalized Cartesian coordinates x_a/r. Thus one can rewrite (12) as

$$f = c^0 + c^1_a \frac{x^a}{r} + c^2_{ab}\frac{x^a x^b - \delta^{ab} r^2}{r^2} + \dots \ , \tag{13}$$

where the coefficients C^l_m in (12) and the symmetric traceless coefficients $c^l_{a_1 \dots a_l}$ in (13) are linearly related. Let J^a denote a $N = m+1$ dimensional representation of the $su(2)$ Lie algebra and define $X^a = fJ^a$ where $f^2 = 4r^2/(N^2-1)$. Then X^a satisfy

$$[X^a, X^b] \ = \ if\epsilon^{abc}X^c \ , \quad X^2 = r^2 \ .$$

We can now define a linear map $\mathcal{M} : S^2_N \longrightarrow \mathrm{Mat}(N)$ from the space of truncated functions (12) to the space of N-dimensional matrices

$$x^{a_1}x^{a_2}\dots x^{a_k} \xrightarrow{\mathcal{M}} \mathrm{sym}(X^{a_1}X^{a_2}\dots X^{a_k})$$

which takes a product of x^a's into the symmetrized product obtained by substituting each x^a by X^a. The (complex) linear dimension of S^2_N is just $\sum_{l=0}^{N-1}(2l+1) = N^2$, the same as the dimension of N-dimensional matrices. In fact the map $\mathcal{M}$ is one to one and onto. Therefore the product of n-dimensional matrices induces a "star" product on the space of truncated functions

$$f_1 * f_2 = \mathcal{M}^{-1}(\mathcal{M}(f_1)\mathcal{M}(f_1)) \ . \tag{14}$$

The $*$-multiplication (14) makes S^2_N into an algebra called the fuzzy sphere algebra.

For practical purposes it is better to use the isomorphism $\mathcal{M}$ and think of S^2_N as the set of N-dimensional matrices with the multiplication given by standard matrix multiplication.

5 Interacting dipoles and the fuzzy sphere

McGreevy, Susskind and Toumbas have performed a classical analysis of the dipole model (9) in [26] and conjectured that the quantum version of the model must be described by noncommutative geometry. Briefly, their analysis is as follows. As one increases the angular momentum of the dipole the distance between the charges also increases. Therefore the angular momentum must be cut off when the size of the dipole equals the diameter of the sphere. It is this angular momentum cutoff which is very suggestive of the fuzzy sphere algebra. In the first part of the paper I have shown that the dipole model [26] gives the low energy description of the open WZW strings. In the remainder of the paper I will derive the fuzzy sphere algebra by studying the interaction vertex of the quantized dipoles thus confirming the McGreevy, Susskind and Toumbas conjecture.

Our first task is to quantize the dipole model (9). It is convenient to add a mass term to the two dipole charges, quantize the model and then take the mass to zero. Then all the higher Landau levels decouple. We will treat the Hamiltonian (11) as a perturbation which will remove the degeneracy of the LLL. Before discussing the dipole I will consider a single charged particle moving on S^2 in a magnetic flux m.

Note that $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ so we can identify functions on S^2 with $\mathrm{U}(1)$ left-invariant functions on $\mathrm{SU}(2)$. Let $\mathcal{D}^j_{pq}(g)$ denote the Wigner symbols defined as

$$\mathcal{D}^j_{pq}(g) = < p\,|U(g)|\,q > \quad , \tag{15}$$

where $|\,q >,\ q = -j, \ldots, j$ are states in the spin j representation of $SU(2)$. For any $h = \mathrm{diag}(e^{i\phi}, e^{-i\phi}) \in \mathrm{U}(1)$ we have

$$\mathcal{D}^j_{pq}(gh) = \mathcal{D}^j_{pq}(g)\ e^{i\,2q\,\phi} \ . \tag{16}$$

Thus, up to normalization we can identify the spherical harmonics Y^p_l with $\mathcal{D}^l_{p0}$ and

expand functions on S^2 as

$$\psi(g) = \sum_{l=0}^{\infty} \sum_{p=-l}^{l} C^l_p \mathcal{D}^l_{p0}(g) \ .$$

The other Wigner symbols $\mathcal{D}^j_{pq}$ for $q \neq 0$ are also useful [27]. It turns out that the functions

$$\psi_m(g) = \sum_{j=q}^{\infty} \sum_{p=-j}^{j} C^j_p \mathcal{D}^j_{pq}(g) \ , \tag{17}$$

where $m = 2q \in \mathbb{Z}$, give the global expansion for sections of the charge m monopole line bundle. From a mathematical point of view (17) is just a useful way of representing section of a nontrivial line bundle as function on a principal bundle. However, it is possible to arrive at this result, as I will briefly discuss by considering the auxiliary problem of a free particle on S^3 followed by a Kaluza-Klein reduction. For the detailed calculations see the appendix.

To understand the above statement, consider a free particle moving on $S^3 = \mathrm{SU}(2)$. The isometry group is SO(4) which up to a global identification is the same as $\mathrm{SU}(2) \times \widetilde{\mathrm{SU}(2)}$. A complete set of commuting generators is given by J_3, $\widetilde{J}_3$ and $J^2 = \widetilde{J}^2$. Then $\mathcal{D}^j_{mn}(g)$ form a complete set of states with eigenvalues m, n, and $j(j+1)$. Locally we can perform a Kaluza-Klein reduction along the U(1) factor. As shown in the appendix, the background metric g_{AB} on S^3 decomposes as follows

$$(g_{AB})_{S^3} \ \longleftrightarrow \ (G_{\mu\nu}, A_\mu, \Phi)_{S^2} \ .$$

Here $G_{\mu\nu}$ is the metric on a S^2 sphere of diameter one, A_μ is the charge one monopole gauge field and Φ gives the size of the U(1) factor.

As usual in Kaluza-Klein reduction, fields carrying momentum in the compact direction are charged under the U(1) gauge fields with charge proportional to the momentum $\widetilde{J}_3$. Therefore, using (16) we see that (17) can be interpreted as the state of a particle of electric charge m in a charge one monopole background. Equivalently we can interpret it as the state of a charge one particle in a m magnetic flux background. The Hamiltonian for the free particle on S^3 is proportional to $J^2 = j(j+1)$ so to restrict (17) to the LLL we must set $j = q$. The reduced Hilbert space is just $\mathcal{H}_m = \mathbb{C}^{m+1}$ and contains only the states

$$\psi_m(g) = \sum_{p=-q}^{q} C_p \mathcal{D}^q_{pq}(g) \ .$$

For a dipole the Hilbert space is given by $\mathcal{H} = \mathcal{H}_m \times \mathcal{H}_{-m} = \mathbb{C}^N \times \mathbb{C}^N$ and this is isomorphic to the space of N-dimensional matrices. This is the first indication that we are on the right track given the relation between the fuzzy sphere algebra and the matrix algebra.[b] What remains to be shown is that matrix multiplication is relevant for describing dipole interactions. First note that states of the dipole are of the form

$$\Psi(g_1, g_2) = \sum_{s,t=-q}^{q} M_{st}\, \mathcal{D}^q_{-s-q}(g_1)\, \mathcal{D}^q_{tq}(g_2)$$

or using the unitarity of the representation

$$\Psi(g_1, g_2) = \sum_{s,t=-q}^{q} M_{st}\, \overline{\mathcal{D}^q_{sq}(g_1)}\, \mathcal{D}^q_{tq}(g_2)\ . \tag{18}$$

It is useful to define an inner product

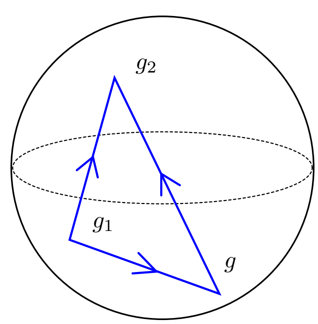
$$< \Psi^{(1)} \,|\, \Psi^{(2)} >= (2q+1)^2 \int\!\!\int_{\mathrm{SU}(2)} dg_1\, dg_2\, \overline{\Psi^{(1)}(g_1, g_2)}\, \Psi^{(2)}(g_1, g_2)\ , \tag{19}$$

where dg_i is the Haar measure on the SU(2) group. Using the orthogonality relations for the Wigner symbols

$$\int_{\mathrm{SU}(2)} dg\, \overline{\mathcal{D}^{j_1}_{p_1 q_1}(g)}\, \mathcal{D}^{j_2}_{p_2 q_2}(g) = \frac{1}{2j_1+1}\, \delta^{j_1 j_2}\, \delta_{p_1 p_2} \delta_{q_1 q_2}\ , \tag{20}$$

we see that $< \Psi^{(1)} \,|\, \Psi^{(2)} >= \mathrm{Tr}(M^{(1)\dagger} M^{(2)})$.

Having performed first quantization for a single dipole we can now consider the Fock space of multi-dipole states and introduce interactions. Since the dipoles are the low energy states of strings it is natural to assume a contact interaction for the endpoints of the dipoles i.e. the positive charge of the first dipole annihilates the negative charge of the second dipole etc. as shown below



[b]If the string stretches between two D-branes with magnetic fluxes m and m' centered at the same point, the Hilbert space is given by $\mathcal{H}_{mm'} = \mathcal{H}_m \times \mathcal{H}_{-m'} = \mathbb{C}^N \times \mathbb{C}^{N'}$ and it is isomorphic to the space of $N \times N'$ dimensional matrices. These are the projective modules of the fuzzy sphere algebra.

For example, the vertex interaction for two incoming and one outgoing dipoles is proportional to

$$
\begin{aligned}
< \Psi^{(3)} \,|\, \Psi^{(1)} * \Psi^{(2)} > \quad &= \\
&(2q+1)^3 \iiint_{\mathrm{SU}(2)} dg_1\, dg_2\, dg_3\, \overline{\Psi^{(3)}(g_1, g_3)} \Psi^{(1)}(g_1, g_2) \Psi^{(2)}(g_2, g_3) \ .
\end{aligned}
$$

The charges of the dipoles have contact interactions and we have to integrate over the locations of the interaction points. Note also that the vertex interaction can be naturally written in terms of the $*$-product

$$
\Psi^{(1)} * \Psi^{(2)}(g_1, g_2) = (2q+1) \int_{\mathcal{SU}(2)} dg\, \Psi^{(1)}(g_1, g)\, \Psi^{(2)}(g, g_2) \ , \tag{21}
$$

and the inner product (19).

Upon integration, using the orthogonality relations for the Wigner symbols (20) the $*$-product (21) reduces to

$$
\Psi^{(1)} * \Psi^{(2)}(g_1, g_2) = \sum_{s,t=-q}^{q} \left(\sum_{u=-q}^{q} M^{(1)}_{su} M^{(2)}_{ut} \right) \overline{\mathcal{D}^q_{sq}(g_1)}\, \mathcal{D}^q_{tq}(g_2) \ .
$$

Lo and behold, the state of the resulting dipole is described by the product $M^{(1)} M^{(2)}$ of the matrices corresponding to the incoming dipoles. Thus I have shown that the vertex interaction of dipoles is naturally described using traces and matrix multiplication, the latter being equivalent to the $*$-multiplication of the fuzzy sphere algebra (14). This is the main result of the paper.

Note that when the location of the two opposite charges of the dipole coincide the wave function is just $\Psi(g, g)$. This is a true function on S^2 and can be identified with the truncation (12). There have been many attempts to write a star product on the fuzzy sphere using (12). In most cases however the resulting formulae lack the simplicity of the Moyal star product on the plane. Alternatively one could use a function of *two* variables $\Psi(g_1, g_2)$ as in (18) in which case the star product has the simple form (21). This remarkable form of representing the fuzzy sphere algebra was introduced in [28] as a trick in an attempt to write the star product. We have found the physical interpretation of (18) as the state of the dipole with the charges on the LLL and of the star product (21) as the contact interaction of dipoles

Finally, note that the Hamiltonian (11) lifts the degeneracy of the states (18). States with different total angular momentum have different energies. It is an interesting exercise to calculate these energies and the propagator.

6 Concluding remarks

While our presentation highlighted the link between the nonlocal nature of the dipole interaction and the fuzzy sphere algebra I would like to point out that we have first taken the low energy limit of the WZW-model and then quantized. In fact one should first quantize and then take the low energy limit as quantization and taking the scaling limit do not commute. Just as for the superstring in flat space (after the GSO projection) the lowest energy states correspond to some of the string oscillator state being excited, the lowest energy states for the WZW open strings also have one oscillator state excited. Since there are three possible oscillators there exist three species of strings (similar to the polarizations states of open strings in flat space). This is why in the matrix model [23] one needs three operators $X^i,\ i = 1, 2, 3$.

Our analysis also makes it clear why it is necessary to restrict to D-branes of finite m magnetic flux. If one takes $k \to \infty$ keeping the ratio m/k fixed it is not necessary to do the rescaling of the coordinate ψ as in (8) and then the kinetic energy term does not vanish. In that case large angular momentum states have energies of the same order of magnitude as stringy excitations. Thus the latter do not decouple. This is the reason for the lack of associativity observed in [18, 29].

A problem left for further study is to extend our approach to D-branes on arbitrary group manifolds.[c] In particular it would be interesting to investigate nonabelian fibers which are related to the nonabelian quantum Hall effect [30].

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Appendix

In this appendix I will discuss the Kaluza-Klein reduction of the Schroedinger equation for a free particle from S^3 = SU(2) to S^2 =SU(2)/U(1) along the U(1) fiber. States

[c] In general these are twined conjugacy classes.

carrying momentum m along U(1) will be identified with a charged particle on S^2 in a magnetic flux m.

In a neighborhood on the identity any $g \in \mathrm{SU}(2)$ can be written as $g(\theta,\phi,y) = g_0(\theta,\phi)h(y)$, where $g_0(\theta,\phi) = e^{i\frac{\sigma_3}{2}\phi}e^{i\frac{\sigma_2}{2}\theta}$ gives a local parameterization of the base and $h(y) = e^{i\sigma_3 y}$ a parameterization of the U(1) fiber. Plugging $g = g_0 h$ into the Killing metric on SU(2) given by $ds^2 = -\frac{1}{2}\mathrm{Tr}(g^{-1}dg\, g^{-1}dg)$ we obtain

$$ds^2 = -\frac{1}{2}\mathrm{Tr}[(g_0^{-1}dg_0 g_0^{-1}dg_0) + 2(g_0^{-1}dg_0 dhh^{-1}) + (h^{-1}dhh^{-1}dh)] \ . \tag{22}$$

On the other hand we can write the metric on S^3 in a Kaluza-Klein form

$$ds^2 = \begin{pmatrix} d\theta & d\phi & dy \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A^T & 1 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \\ dy \end{pmatrix} , \tag{23}$$

where $G_{\mu\nu}$ is the metric on the S^2 base space, A_μ is a U(1) gauge field and Φ is a scalar. We can obtain $G_{\mu\nu}$, A_μ and Φ by comparing (22) and (23). After a trivial calculation we have

$$G_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad A_\mu dx^\mu \ = \frac{1}{2}\cos\theta d\phi \ , \quad \Phi \ = 1 \ ,$$

that is, the metric on a S^2 of diameter one, the monopole gauge field and a constant scalar field measuring the radius of the $U(1)$ fiber. Note that g_0 is not well defined globally. However we can cover the base with patches to obtain the global description. Locally different choices of g_0 are then $U(1)$ gauge equivalent.

Next consider a (non relativistic) scalar field on S^3 with the action given by

$$\mathcal{S} = \int dt \int d^3x\sqrt{g}\left(i\overline{\Psi}\partial_t\Psi - \frac{1}{2M}\,\partial_A\overline{\Psi}g^{AB}(x)\partial_B\Psi \,-\, E_0\overline{\Psi}\Psi\right) \ , \tag{24}$$

where g_{AB} is the standard S^3 metric. The equation of motion derived from (24) is just the Schroedinger equation for a free particle of mass M, and the last term is such that the ground state energy is set to E_0.

The isometry group is SO(4) which modulo some global identifications is just $\mathrm{SU}(2)\times \widetilde{\mathrm{SU}(2)}$. A complete set of commuting operators is given by J_3, $\widetilde{J}_3$ and $J^2 = \widetilde{J}^2$. Then the Wigner symbols (15) form a complete set of states and satisfy

$$\begin{aligned} J_3\,\mathcal{D}^j_{pq}(g) &= p\,\mathcal{D}^j_{pq}(g) \ , \\ \widetilde{J}_3\,\mathcal{D}^j_{pq}(g) &= q\,\mathcal{D}^j_{pq}(g) \ , \\ J^2\,\mathcal{D}^j_{pq}(g) &= j(j+1)\,\mathcal{D}^j_{pq}(g) \ . \end{aligned}$$

Furthermore, the Hamiltonian which up to an additive constant is proportional to the Lapacian, can be written as $\mathcal{H}_{S^3} = \frac{2}{M} J^a J^a + E_0 = \frac{2}{M} j(j+1) + E_0$.

Locally we can perform the Kaluza-Klein reduction along the U(1) factor. If we substitute

$$\Psi(\theta, \phi, y) = \sum_{m \in \mathbb{Z}} \Psi_m(\theta, \phi) \frac{e^{imy}}{\sqrt{2\pi}}$$

and the decomposition (23) of the metric into the action (24) we obtain $\mathcal{S} = \sum_{m \in \mathbb{Z}} \mathcal{S}_m$ where

$$\begin{aligned} \mathcal{S}_m &= \int dt \int d\theta d\phi \sqrt{G} \left[i \overline{\Psi}_m \partial_t \Psi_m \right. \\ &- \left. \frac{1}{2\mathcal{M}} \overline{D_\mu^{(m)} \Psi_m} G^{\mu\nu} D_\mu^{(m)} \Psi_m - \left(\frac{1}{2M} m^2 + E_0 \right) \overline{\Psi}_m \Psi_m \right] . \end{aligned} \tag{25}$$

Here $D_\mu^{(m)} = \partial_\mu - imA_\mu$ is the covariant derivative in a charge m monopole background. Thus fields carrying momentum m along the compact direction are charged under the U(1) gauge field with charge m .

Let us concentrate on the momentum $m = 2q$ sector for which a complete set of eigenfunctions is given by $\mathcal{D}^j_{pq}(g)$. Using (25) we see that the Hamiltonian $\mathcal{H}_{S^3}$ is related to the Hamiltonian $\mathcal{H}^{(m)}_{S^2}$ of the particle on S^2 in a flux m magnetic field by $\mathcal{H}_{S^3} = \mathcal{H}^{(m)}_{S^2} + \frac{1}{2M} M^2 + E_0$. If we set $E_0 = \frac{m}{M}$ the eigenfunctions $D^j_{pq}(g)$ satisfy

$$\mathcal{H}^{(M)}_{S^2} \mathcal{D}^j_{pq}(g) = \frac{2}{M} [(j(j+1) - q(q+1)] \mathcal{D}^j_{pq}(g) . \tag{26}$$

Thus the lowest LLL $\mathcal{D}^q_{pq}(g)$, $p = -q, \ldots, q$ obtained for $j = q$ have zero energy. As can be seen from (26) in the limit $M \to \infty$ all the higher Landau levels decouple.

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